Some Recent Advances on Steiner Trees and Related Problems

Siavash Vahdati Daneshmand
Tobias Polzin

Institut für Informatik, Universität Mannheim
Max-Planck-Institut für Informatik, Saarbrücken
Recent Advances on Steiner Trees

Outline

• Steiner tree problem
  – basics
  – FST approach

• minimum spanning tree in hypergraph
  – comparison of relaxations

• recent successful techniques
  – extended reduction methods
  – use of low connectivity
  – local generation of cutting planes
The Steiner Tree Problem in Networks

- **instance:**
  - network $G=(V,E,c)$
  - terminals $R$

- **solution:**
  - subnetwork of $G$ spanning $R$
  - minimum cost
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SMT: Steiner Minimal Tree
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Applications, Theoretical Results

- fundamental network design problem
- applications:
  - routing
  - pipeline-planning
  - VLSI-design
  - phylogenetic trees
- theoretical results:
  - NP-hard, APX-complete
  - best known approximation algorithm:
    performance ratio $\approx 1.55$
(Di-)Cut Formulation

- (bidirected) arc set $A$
- Choose root $z_1 \in R$, $R_1 = R - z_1$
- for each terminal $z \in R_1$, send one unit of (corresponding) flow from root to $z$

$$\begin{align*}
P_C \quad & \sum_{a \in A} c_a y_a \to \min, \\
& \sum_{a \in \delta^{-}(S)} y_a \geq 1 \quad (z_1 \notin S, \ S \cap R_1 \neq \emptyset), \\
& y_a \in \{0, 1\} \quad (a \in A).
\end{align*}$$
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(Dis-)Cut Relaxation

- **flow/cut relaxation:**
  - relaxing integrality constraints
  - solution: rational vector not (arcwise) smaller than the flow vectors (Example: $v(LP_c) = 7.5$)

- **integrality gap:**
  - worst known case: 8/7
  - current guarantee: 2
    (3/2 for quasi-bipartite graphs)

Linear solution (blue and green flows 0.5, value 7.5)
FST Method for (Geometric) Steiner Problems

• Full Steiner Tree (FST): tree with no inner terminal

• FST generation: set $F$ of FSTs which contains an SMT

• FST concatenation: subset of $F$ whose concatenation is an SMT
MSTH Approach for FST Concatenation

- Consider terminal sets of FSTs as edges of a hypergraph
- Find a minimum spanning tree in hypergraph (MSTH)
- MSTH problem is NP-hard
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**MSTH: Packing Formulation**

$$
\begin{align*}
\min \quad & \sum_{T \in F} c_T X_T \\
\text{s.t.} \quad & \sum_{T \in F} (|T| - 1) X_T = |R| - 1, \\
& \sum_{T, T \cap S \neq \emptyset} (|T \cap S| - 1) X_T \leq |S| - 1 \quad (\emptyset \neq S \subseteq R), \\
& X_T \in \{0, 1\} \quad (T \in F).
\end{align*}
$$
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**MSTH: Packing Formulation, Directed**

\[ P_{\text{FST}} \sum_{\vec{T} \in \vec{F}} c_{\vec{T}} x_{\vec{T}} \to \min, \]

\[ \sum_{\vec{T} \in \vec{F}} (|\vec{T}| - 1)x_{\vec{T}} = |R| - 1, \]

\[ \sum_{\vec{T}, \vec{T} \in \Delta^-(z_i)} x_{\vec{T}} = 1 \quad (z_l \in R_1), \]

\[ \sum_{\vec{T}, \vec{T} \cap S \neq \emptyset} (|\vec{T} \cap S| - 1)x_{\vec{T}} \leq |S| - 1 \quad (\emptyset \neq S \subset R), \]

\[ x_{\vec{T}} \in \{0, 1\} \quad (\vec{T} \in \vec{F}). \]
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**MSTH: Packing Formulation, Directed**

\[
\begin{align*}
\text{min,} & & \sum_{\bar{T} \in \bar{F}} c_{\bar{T}} x_{\bar{T}} \\
\text{s.t.} & & \sum_{\bar{T} \in \bar{F}} (|\bar{T}| - 1) x_{\bar{T}} = |R| - 1, \\
& & \sum_{\bar{T}, \bar{T} \in \Delta^{-}(z_i)} x_{\bar{T}} = 1 \quad (z \in R), \\
& & \sum_{\bar{T}, \bar{T} \cap S \neq \emptyset} (|\bar{T} \cap S| - 1) x_{\bar{T}} \leq |S| - 1 \quad (\emptyset \neq S \subset R), \\
& & x_{\bar{T}} \in \{0, 1\} \quad (\bar{T} \in \bar{F}).
\end{align*}
\]

\( LP_{FST} \rightarrow \) is equivalent to \( LP_{FST} \)
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MSTH: Directed Cut Formulation

\[ P_{FSC} \quad \sum_{\bar{T} \in \bar{F}} c_{\bar{T}} x_{\bar{T}} \rightarrow \min, \]
\[ \sum_{\bar{T}, \bar{T} \in \Delta^-(S)} x_{\bar{T}} \geq 1 \quad (z_1 \notin S, \ S \cap R_1 \neq \emptyset), \]
\[ x_{\bar{T}} \in \{0, 1\} \quad (\bar{T} \in \bar{F}). \]

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MSTH: Directed Cut Formulation

\[ \begin{align*}
P_{FSC} & \quad \sum_{\tilde{T} \in \tilde{F}} c_{\tilde{T}} x_{\tilde{T}} \rightarrow \min, \\
& \quad \sum_{\tilde{T}, \tilde{T} \in \Delta^{-}(S)} x_{\tilde{T}} \geq 1 \quad (z_1 \notin S, \ S \cap R_1 \neq \emptyset), \\
& \quad x_{\tilde{T}} \in \{0, 1\} \quad (\tilde{T} \in \tilde{F}).
\end{align*} \]

\[ \text{LP}_{FSC} \text{ is equivalent to } \text{LP}_{FST} \]

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Comparison of MSTH and SMT Relaxations

• worst case perspective:
  – \( LP_{FSC} \) is (strictly) stronger than \( LP_{C} \)
  – Note: \( \nu(LP_{FSC}) \) depends on the choice of FSTs

• empirical:
  – Both methods usually yield the same value

<table>
<thead>
<tr>
<th>instance group</th>
<th>( LP_{FST} )</th>
<th>( LP_{C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>gap (%)</td>
<td>time (s)</td>
</tr>
<tr>
<td>ES1000FST</td>
<td>0.0078</td>
<td>99.2</td>
</tr>
<tr>
<td>TSPFST</td>
<td>0.009803</td>
<td>129.6</td>
</tr>
</tbody>
</table>
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**Reductions: Extending the Scope**

- **basic idea:** extending the scope beyond single vertices, edges
- **our approach:** combined use of alternative- and bound-based methods
  - alternative-based: cheaper MST/SMT for leaves with respect to Steiner bottleneck distances
  - bound-based: lower bound above upper bound for all orientations
  - complementary strength: different combinations excluded with different methods
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Low Connectivity

- vertex separator $S \subset R$
- SMT $T$ (unknown)
- $T$ restricted to $G_i$: forest $F_1$
- $F_1$ yields a partition of $S$
- number of possible partitions:

$$B(|S|) = \sum_{i=1}^{\lfloor |S|/i \rfloor} \left\lfloor \frac{|S|}{i} \right\rfloor$$

| $|S|$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|
| cases | 2 | 5 | 15 | 52 | 203 | 877 | 4140 |
Local Approach: Reduction by Partitioning

- Compute (many) small vertex separators
  - up to $\Theta(|R|)$ separators of size at most $k$ in time $O(k|R||E|)$
- Exact method:
  - compute optimum forests in $G_2$ for all cases
    (many cases can be ruled out by heuristics)
  - Contract common edges, delete unused edges
- Bound-based method:
  - Extend $G_2$ to $G'_2,G''_2$ using information from $G_1$
  - Compute upper bound in $G'_2$
  - Compute lower bound in $G''_2$ under some constraint
    (e.g., that a vertex must be included in the solution)
  - Constraint cannot be (optimally) satisfied if
    \[ upper(G'_2) < lower_{constrained}(G''_2) \]
Global Approach: Dynamic Programming

- **bottom-up algorithm:**
  - choose a vertex $v_1$ as the first visited vertex
  - for $s = 2,...,|V|$
    - choose unvisited $v_s$ adjacent to some visited vertex
    - for all currently valid partitions $P$ in the border
      - for all choices $I$ of sets in $P$ adjacent to $v_s$
        - merge the sets in $I$ via $v_s$ at minimum cost, update $P$ to $P'$
        - if $|P'|=1$ check whether a better Steiner tree is found

border after step $s$: $B_s = \{ v_i \in \{v_1,...,v_s \} | \exists (v_i,v_j) \in E : v_j \notin \{v_1,...,v_s \} \}$
max$\{|B_s| | 1 \leq s \leq |V|\}$ = $b \leftrightarrow$ path decomposition of $G$ of width $b$
running time: $|V|2^{O(b \log b)}$
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Local Generation of Cutting Planes

- find optimum solution $x^*$ of current LP
- find suitable components $V_i$ in $(V, \{e \in E \mid x_e^* > 0\})$
- for all $V_i$ of proper size:
  - shrink $G$, $x^*$ to $G_1$, $x_1^*$ by contracting all $V_j$, $j \neq i$
- initialize matrix $T$ with (the incidence vector of) a Steiner tree for $G$
- repeat
  - $a^* = \arg \min \{x^* a \mid Ta \geq 1, \ a \geq 0\}$
  - if $x^* a^* \geq 1$ break
  - find SMT $t^*$ in $G$ w.r.t. cost vector $a^*$
  - if $t^* a^* < 1$, add $t^*$ to $T$
  - else
    - expand $a^*$ to $a^*$ in $G$
    - add $xa^* \geq 1$ to LP
    - break
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FL1400fst (after FST generation)

|V| = 2694  |E| = 4546  |R| = 1400

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FL1400fst: after some reductions

$|V|=1871$  $|E|=3474$  $|R|=937$

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**FL1400fst: reduced instance**

\[ |V| = 1871 \quad |E| = 3474 \quad |R| = 937 \]

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**FL1400fst: a component**

|V| = 427  \quad |E| = 796  \quad |R| = 215  \quad b = 9

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ES10000: the terminals

$|R| = 10000$

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ES10000: after FST generation

|V| = 27019  |E| = 39407  |R| = 10000

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ES10000fst: after some reductions

$|V| = 10865 \quad |E| = 17764 \quad |R| = 4322$
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ES10000fst: reduced instance

|V| = 10865  |E| = 17764  |R| = 4322

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**ES10000fst: after reduction by partitioning**

\[ |V| = 4197 \quad |E| = 6927 \quad |R| = 1614 \]